In other words, we have shown that if a face of our tetrahedron has a nonacute angle at one vertex, the face of the tetrahedron opposite that vertex must be an acute triangle. It follows that for our tetrahedron to have three nonacute face angles, they would necessarily have C as a vertex. But it is easily seen that the circumcentre of a tetrahedron having three nonacute angles at C would necessarily be exterior to the tetrahedron: Returning to the figure, we see that the line through C perpendicular to the plane BCD (and therefore parallel to OO_1) would intersect the sphere at a point, call it P that is separated from the vertex A by the plane through BD that is parallel to CP. The faces BCP and DCP of the tetrahedron $PBCD$ have right angles at P. To make those angles obtuse, P would have to be chosen on the sphere further from A; that is, at least one of the angles BCA and DCA would have to be acute.

3671★. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $ABCD$ be a tetrahedron and let M be a point in its interior. Prove or disprove that

$$\frac{[BCD]}{AM^2} = \frac{[ACD]}{BM^2} = \frac{[ABD]}{CM^2} = \frac{[ABC]}{DM^2} = \frac{2}{\sqrt{3}},$$

if and only if the tetrahedron is regular and M is its centroid. Here $[T]$ denotes the area of T.

No solutions have been received. This problem remains open.

3672. [2011 : 389, 392] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x and y be real numbers such that $x^2 + y^2 = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy} \ \geq \ \frac{3}{1+\left(\frac{x+y}{2}\right)^2}.$$

When does this inequality occur?

Solution by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Marian Dincă, Bucharest, Romania; Dimitrios Koukakis, Kato Apostoloi, Greece; Kee-Wai Lau, Hong Kong, China; Salem Malikić, student, Simon Fraser University, Burnaby, BC; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comăneşti, Romania; and the proposer.

Let $t = xy$. Since $2|xy| \leq x^2 + y^2 = 1$, $|t| \leq \frac{1}{2}$. The difference between the two sides of the proposed inequality is

$$\begin{aligned}\frac{2+x^2+y^2}{1+x^2+y^2+x^2y^2} + \frac{1}{1+xy} - \frac{12}{4+x^2+y^2+2xy} \\ &= \frac{3}{2+t^2} + \frac{1}{1+t} - \frac{12}{5+2t} \\ &= \frac{(1-2t)(1+3t+5t^2)}{(2+t^2)(1+t)(5+2t)} = \frac{(1-2t)(1+t^2+(1+3t)^2)}{2(2+t^2)(1+t)(5+2t)} \\ &\geq 0\end{aligned}$$

with equality if and only if t $= 1/2$. With the given condition, this implies that equality occurs if and only if $x = y = \pm 1/\sqrt{2}$.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; STAN WAGON, Macalester College, St. Paul, MN, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA;

Wagon used mathematical software to find that, when $x^2+y^2=1$,

$$3 \leq \left(\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy}\right)\left(1 + \left(\frac{x+y}{2}\right)^2\right) \leq \frac{10}{3}$$

with equality on the left if and only if $x = y = \pm 1/\sqrt{2}$. and equality on the right if and only if $x = -y = \pm 1/\sqrt{2}$.

3673. [2011 : 390, 392] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the product

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)^{(-1)^{n-1}} .$$

I. Composite of solutions by Paul Bracken, University of Texas, Edinburg, TX, USA; Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA; Dimitrios Koukakis, Kato Apostoloi, Greece; and AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

The answer is $\pi^2/8$. Recall the Wallis formula:

$$\lim_{m\to\infty} \frac{1}{2m+1} \prod_{k=1}^{m} \frac{(2k)^2}{(2k-1)^2} = \frac{\pi}{2}.$$

For $n \geq 2$, let

$$P(n) = \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)^{(-1)^{k-1}} .$$